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Differential equations - Introduction

A **differential equation** is an equation involving a variable and its derivatives with respect to one or more independent variables. Differential equations often arise in modelling real world phenomena — derivatives give rates of change, and rates of change are often empirically measurable.

The **order** of the equation is the order of the highest-order derivative that it contains. If there is a single independent variable, the equation is an **ordinary differential equation (ODE)**; if there are several independent variables, it is a **partial differential equation (PDE)**.

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 e^x \qquad \text{first order, ordinary}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{second order, partial}$$

To **solve** the differential equation means (roughly) to find an expression for the dependent variable in terms of the independent variables which satisfies the original equation.

Example. $\dfrac{dy}{dx} = 2(1 - y^2)$

The solution is

$$y = \frac{ce^x - 1}{ce^x + 1}$$

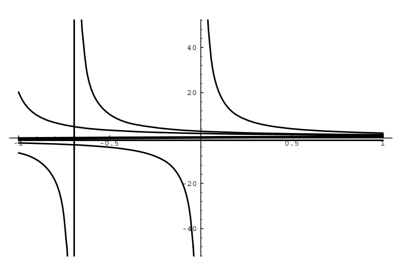
c is an **arbitrary constant**. That is, the expression above is a solution for any value of c: $c = 1$, $c = \pi$, $c = -7.9$, and so on.

You can verify that $y = \dfrac{ce^x - 1}{ce^x + 1}$ solves the equation by plugging it into both sides and checking that the equation is true:

$$\frac{dy}{dx} = \frac{2ce^x}{(ce^x + 1)^2}, \quad \frac{1}{2}(1 - y^2) = \frac{2ce^x}{(ce^x + 1)^2}.$$

It is good to remember that you can check the solution to a differential equation by plugging in.

Note that each value of c gives a different solution $y = \dfrac{ce^x - 1}{ce^x + 1}$. Intuitively, the original equation involves a first derivative. You "undo" a *first* derivative by integrating *once*. A *single* integration produces *one* arbitrary constant.



The picture shows the solution curves for $c = -3, -2, -1, 0, 1, 2, 3$. The solution curves for different values of c form a family of curves which fill up the plane. They may remind you of the **integral curves** of a vector field. Indeed, the two situations are closely related. $\square$

Take a first order equation

$$\frac{dy}{dx} = f(x, y).$$

$\dfrac{dy}{dx}$ is the slope of a solution curve, so the equation says that $f(x, y)$ is the slope of a solution curve at the point (x, y). For example, suppose

$$\frac{dy}{dx} = \frac{x}{y+1}.$$

The slope of the solution curve passing through the point $(4, 1)$ is $\dfrac{dy}{dx} = \dfrac{4}{1+1} = 2$.

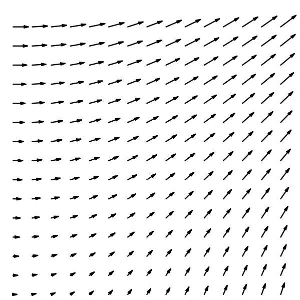
It follows that you can get a rough picture of the solution curves by drawing a little segment at each point (x, y) such that the slope of the segment is $f(x, y)$. You could do this by hand with a piece of graph paper; you can also use q computer equipped with the appropriate software. The symbolic math package *Mathematica* has a function called `PlotVectorField` which draws a picture of a vector field. Here's how to use it.

First, you will need to load the package containing the function:

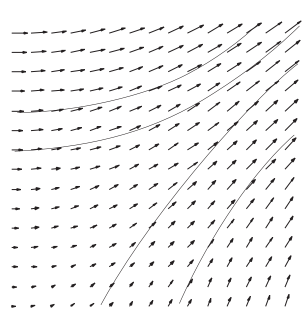
```
Needs["Graphics`PlotField`"]
```

I'll use $\dfrac{dy}{dx} = \dfrac{x}{y+1}$ as an example. Think of the fraction as dy divided by dx, with $dy = x$ and $dx = y + 1$. The vector field is $\langle y + 1, x\rangle$. The following command draws a picture of the field:

```
PlotVectorField[{y + 1, x}, {x, 0, 3}, {y, 0, 3}]
```

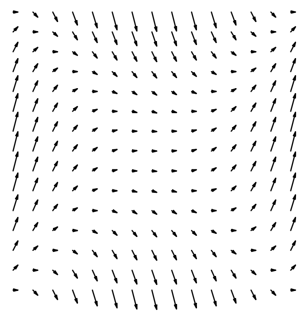


You can get the solution curves by sketching in curves which follow the arrows:



What do you do with something like $\dfrac{dy}{dx} = x^2 - y^2$? It isn't obviously a fraction. Just choose dx and dy so the quotient is $x^2 - y^2$. For example, $dx = 1$ and $dy = x^2 - y^2$ will work:

```
PlotVectorField[{1, x^2 - y^2}, {x, -2, 2}, {y, -2, 2}]
```



The pictures above are called **direction fields**. Note that you can draw them *without* actually solving the equation. Hence, you can sometimes tell things about the solution curves without actually solving the equation.

Generically, the **general solution** to an n-th order differential equation has n arbitrary constants. To put things informally, the general solution is an expression which contains all possible solutions as special cases.

This course is primiarly concerned with **ordinary differential equations**. **Partial differential equations** are often more difficult to solve, and may require techniques such as **Fourier series**.

Example. Verify that $u = x^2 + t^2$ is a solution to

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

(This equation is a special case of the **wave equation**.)

$$u_{xx} = 2 = u_{tt}. \quad \square$$

Example. Find the values of r such that $y = e^{rx}$ is a solution to

$$y'' - 2y' - 3y = 0.$$

(The derivatives are taken with respect to x.)

Compute the first and second derivatives:

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

Plug them into the differential equation and solve for r:

$$y'' - 2y' - 3y = r^2 e^{rx} - 2re^{rx} - 3e^{rx} = (r^2 - 2r - 3)e^{rx} = 0.$$

Then $r^2 - 2r - 3 = 0$, or $(r-3)(r+1) = 0$, so $r = 3$ or $r = -1$.

e^{3x} and e^{-x} are solutions to the equation. $\square$

Remarks.

1. The previous example shows that if you can guess the *form* of a solution to a differential equation, you can often obtain a solution.

2. An equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

is a **linear equation** in y. The dependent variable y and its derivatives only occur to the first power, with coefficients which are functions of x alone. $\square$